Question:

① Using the definition, find the derivative of $y = \frac{1}{x}$ at any point x.

② Using your answer, find all points of the graph $y = \frac{1}{x}$ when the tangent line is parallel to $y = -2x + 2489$

$= mx + b$

① $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\frac{1}{(x+h)} - \frac{1}{(x)}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h \to 0} \dfrac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim\limits_{h \to 0} \dfrac{\frac{x-x-h}{x(x+h)}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\frac{-h}{x(x+h)}}{h/1} = \lim\limits_{h \to 0} \dfrac{-\cancel{h} \cdot 1}{x(x+h) \cdot \cancel{h}}$

$= \lim\limits_{h \to 0} \dfrac{-1}{x(x+h)} = \boxed{\dfrac{-1}{x^2}}$

(2) $-2 = \frac{-1}{x^2}$

(slope of the line) (slope of tangent line of the fcn.)

$-2x^2 = -1$

$x^2 = \frac{-1}{-2} = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}}$ $\quad x = \pm\left(\frac{1}{2}\right)^{1/2}$

$= \pm\frac{\sqrt{1}}{\sqrt{2}}$

$x = \pm\frac{1}{\sqrt{2}}.$

$\boxed{\text{points } \left(+\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)}$

$x, \frac{1}{x} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

$x, \frac{1}{x} = \frac{1}{-1/\sqrt{2}}$

$= -\sqrt{2}$

Theorem (Power Rule) $(x^n)' = n \cdot x^{n-1}$.

Proof: Let $F(x) = x^n$ — n fixed.

I will assume it is a positive integer & prove in that case.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \leftarrow$$

Question: What is $(x+h)^n$?

Pascal's triangle:

$$\begin{array}{l} 1 \\ 1 \quad 1 \\ (x+h)^2 \to 1 \quad 2 \quad 1 \\ (x+h)^3 \to 1 \quad 3 \quad 3 \quad 1 \\ (x+h)^4 \to 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ (x+h)^5 \to 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \vdots \end{array}$$

Then

$$(x+h)^n = x^n + n x^{n-1} \cdot h + (\text{blah})\, x^{n-2} h^2 + \begin{pmatrix}\text{blah} \\ \text{blah}\end{pmatrix} x^{n-3} h^3 + \dots$$

Actually, in the $n^{\underline{th}}$ row of Pascal's Triangle:

$$1 \quad n \quad \frac{n(n-1)}{2} \quad \frac{n(n-1)(n-2)}{3\cdot 2} \quad \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2} \quad \cdots$$

(blah) (blah blah) ~ ~ ~

$$\therefore\ F'(x) = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + (\text{blah})x^{n-2}h^2 + \cdots - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + (\text{blah})x^{n-2}h^2 + \cdots}{h}$$

all have h

$$= \lim_{h\to 0} \frac{\cancel{h}(nx^{n-1} + (\text{blah})x^{n-2}h^1 + \cdots)}{\cancel{h}}$$

$$= \boxed{nx^{n-1}}$$

$\blacksquare$

Let's use the power rule:

- Find the slope of the tangent line to $y = \frac{1}{\sqrt[3]{x}}$ at $x = 1$.

Solution: $y = \frac{1}{x^{4/3}} = x^{-1/3}$

$$y' = nx^{n-1} = \left(\frac{-1}{3}\right) \cdot x^{\left(\frac{-1}{3}\right)-1}$$

$$= -\frac{1}{3}x^{-4/3} = \frac{-1}{3x^{4/3}}$$

$x=1 \Rightarrow \quad y' = \boxed{\frac{-1}{3}}$.

Properties of the Derivative.

(1) Constant multiples

If $g(x) = c \cdot f(x)$ for some constant c, then

$$g'(x) = c \cdot f'(x).$$

Proof:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$= \lim_{h\to 0} c\cdot\frac{(f(x+h)-f(x))}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = c\cdot f'(x).$$

② Derivative of a constant.

If $F(x) = K$ for some constant K,

Then $F'(x) = 0$.

Proof: graph is a horizontal line $\Rightarrow$ slope $= 0$ everywhere $\Rightarrow F'(x) = 0$.

③ Sums & Differences of functions.

Theorem: If $B(x) = f(x) + g(x)$, then $B'(x) = f'(x) + g'(x)$.

Also, if $A(x) = f(x) - g(x)$

then $A'(x) = f'(x) - g'(x)$.

Proof for $B(x)$:

$$B'(x) = \lim_{h \to 0} \frac{B(x+h) - B(x)}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

Commutative property of addition

$A + B = B + A$

Definition of subtraction

$A - B = A + {}^{-}B$

$$= f'(x) + g'(x). \ \square$$

<u>Example</u> If $\alpha(x) = \dfrac{\sqrt{x} + 7x}{x^3}$,

find $\alpha'(x)$.

$$\alpha'(x) = \left(\frac{x^{1/2} + 7x}{x^3}\right)'$$

$$= \left[(x^{1/2} + 7x)x^{-3}\right]'$$

$$= \left[x^{\frac{1}{2}}x^{-3} + 7x \cdot x^{-3}\right]'$$

$$= \left[x^{-\frac{5}{2}} + 7x^{-2}\right]'$$

$$= \left[x^{-5/2}\right]' + \left[7x^{-2}\right]'$$

by Sum Rule

$$= \left[x^{-5/2}\right]' + 7 \cdot \left[x^{-2}\right]'$$

by constant multiple rule

Power Rule $= \left(\frac{-5}{2}\right)\cdot x^{-7/2} + 7\cdot(-2)x^{-3}$

$$\alpha'(x) = \boxed{-\frac{5}{2}x^{-7/2} - 14x^{-3}}.$$

Let's derive (prove) some more formulas for the derivative of functions.

① Find the formula for the derivative of $\sin(\theta) = \omega(\theta)$.

$$\frac{d\omega}{d\theta} = \omega'(\theta) = \lim_{h\to 0} \frac{\omega(\theta+h) - \omega(\theta)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(\theta+h) - \sin(\theta)}{h}$$

$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$

$$= \lim_{h\to 0} \frac{\sin(\theta)\cos(h) + \sin(h)\cos(\theta) - \sin(\theta)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin(\theta)\cos(h) - \sin(\theta)}{h} + \underbrace{\frac{\sin(h)}{h}}_{\to 1}\cos(\theta) \right)$$

$$= \lim_{h\to 0} \frac{\sin(\theta)(\cos h - 1)}{h} + \cos(\theta)$$

$$= \lim_{h\to 0} \frac{\sin(\theta)(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)} + \cos(\theta)$$

$$= \lim_{h\to 0} \frac{\sin(\theta)(\cos^2(h)-1)}{h(\cos(h)+1)} + \cos(\theta)$$

$$(A-B)(A+B) = A^2 - B^2$$

$$= \lim_{h\to 0} \frac{\sin(\theta)(-\sin^2(h))}{h(\cos(h)+1)} + \cos(\theta)$$

$$\cos^2(h) + \sin^2(h) = 1$$
$$\cos^2(h) - 1 = -\sin^2(h)$$

$$= \lim_{h\to 0} - \underbrace{\frac{\sin(h)}{h}}_{1} \frac{\sin(\theta)\overbrace{\sin(h)}^{0}}{\underbrace{(\cos(h)+1)}_{2}} + \cos(\theta)$$

$$= -\frac{1 \cdot \sin(\theta) \cdot 0}{2} + \cos(\theta)$$

$$= 0 + \cos\theta = \boxed{\cos(\theta)}$$

$$\therefore (\sin(\theta))' = \cos\theta.$$

$\blacksquare$

In homework: you will prove
$$(\cos(\theta))' = -\sin(\theta).$$

<u>Fact</u>. If $\phi(x) = e^x$, then
$$\phi'(x) = e^x.$$

<u>Proof:</u>
$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$
$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$
$$= \boxed{e^x}.$$

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Quiz

1. What is your name?
2. What is your second favorite book?
3. Complete these identities
   a. $\sin(2\theta) =$ ______________
   b. $\sin^2(\theta) =$ ______________
4. Simplify:

$$e^4\left(e^{2x}\right)^3$$

5. Find the slope of the tangent line to $y = x^2$ at $x = 2$, using the definition.